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A VARIANT OF YOUNG'S METHOD IN VOTING THEORY PROVIDING THE SAME WINNERS AS COPELAND'S METHOD

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ABSTRACT

In the framework of voting theory, Young's method consists in removing a minimum number of voters in order to obtain a Condorcet winner. We study here another method, consisting in removing a minimum number of candidates in order to obtain a Condorcet winner. We show that this method leads to the same winners as Copeland's tournament solution, which selects a candidate who defeats a maximum number of other candidates in the pairwise comparison method advocated by Condorcet.

Keywords: Social choice, elections, voting theory, tournament solutions, pairwise comparison method, Condorcet winner, Young's method, Copeland's solution, removal of voters, removal of candidates.

I. INTRODUCTION

At the end of the 18th century, Condorcet [5] suggested a method, based on pairwise comparisons, in order to determine the winner of an election[2]. More precisely, a *Condorcet winner* is a candidate who is preferred to any other candidate by a majority of voters. Unfortunately, as shown by Condorcet himself, such a candidate does not always exist. Several methods have been designed in order to restore a Condorcet winner when the preferences of the voters do not generate such a Condorcet winner. Among them, Young's method [15] (see also [4]) consists in removing a minimum number of voters so that a Condorcet winner appears. From a democratic point of view, it can be quite difficult to explain to the voters why it is necessary not to take their preferences into account and not the votes of some other voters. Moreover, the computation of the Young winners is difficult, more precisely NP-hard [14], as well as checking that a given candidate is a Young winner. It is maybe more realistic to imagine that agreements or negotiations may lead some candidates to withdraw their candidatures. We can then wonder how many candidates it would be necessary to remove in order to obtain a new election admitting a Condorcet winner.

In this paper, I investigate such an approach and I show that removing a minimum number of candidates in order to obtain a Condorcet winner leads to the same winners as another well-known method, namely Copeland's method [6][3], which selects a candidate who defeats a maximum number of other candidates in the pairwise comparison method. This constitutes, to my knowledge, a new property of Copeland's method.

More precisely, Section II depicts Copeland's method and Young's method, within the context of voting theory. Section III specifies the new property of Copeland's solution.

II. COPELAND'S TOURNAMENT SOLUTION AND YOUNG'S METHOD

In order to determine the winner of an election, Condorcet [5] designed the following method, based on pairwise comparisons: for each candidate x and each candidate y with $x \neq y$, we compute the number m_{xy} of voters who prefer x to y. Then x is considered as preferred to y in Condorcet's method if a majority of voters prefers x to y, *i.e.* if we have $m_{xy} > m_{yx}$. This defines the (*strict*) *majority relation* T: $xTy \Leftrightarrow m_{xy} > m_{yx}$. In some cases, there exists a *Condorcet winner*, *i.e.* a candidate C defeating any other candidate in such a pairwise comparisons method: $\forall\ x \neq C$, $m_{Cx} > m_{xC}$. If there exists a Condorcet winner, then he or she is unique and several voting procedures then select this Condorcet winner as the winner of the election. But it is well-known, as shown by Condorcet himself, that this method may fail in determining a winner, even if all the voters' preferences are assumed to be linear orders. Such a situation can be illustrated by the following example.

Example: Assume that $m = 7$ voters must rank $n = 4$ candidates a, b, c, and d. The preferences of the voters are supposed to be given by the following linear orders, where $x > y$ means that x is preferred to y by the considered voter (with the transitivity assumption: $x > y$ and $y > z$ yield $x > z$):

- the preferences of two voters are: $a > b > c > d$;
- the preferences of two voters are: $c > d > a > b$;
- the preference of one voter is: $b > d > a > c$;
- the preference of one voter is: $c > b > d > a$;
- the preference of one voter is: $d > a > b > c$.

The quantities m_{xy} involved in Condorcet's procedure are the following, where the bold values show, for each pair $\{x, y\}$ of candidates with $x \neq y$, the largest between the two quantities m_{xy} and m_{yx}:

- $\boldsymbol{m_{ab} = 5}$; $m_{ba} = 2$;
- $\boldsymbol{m_{ac} = 4}$; $m_{ca} = 3$;
- $m_{ad} = 2$; $\boldsymbol{m_{da} = 5}$;
- $\boldsymbol{m_{bc} = 4}$; $m_{cb} = 3$;
- $\boldsymbol{m_{bd} = 4}$; $m_{db} = 3$;
- $\boldsymbol{m_{cd} = 5}$; $m_{dc} = 2$.

Here, there is no Condorcet winner. The majority relation T_{Ex} for the example is given by: $aT_{Ex}b$, $aT_{Ex}c$, $bT_{Ex}c$, $bT_{Ex}d$, $cT_{Ex}d$, $dT_{Ex}a$ (T_{Ex} is not transitive).

If we assume that there is no tie (*i.e.*, we cannot have $m_{xy} = m_{yx}$), as we do in the sequel, the majority relation T is a *tournament*, *i.e.* an antisymmetric and complete relation: for any pair of distinct candidates $\{x, y\}$, one and only one of the two possibilities xTy or yTx occurs.

Let X denote the set of candidates. In his book [9], J.-F. Laslier defines a *tournament solution* S as any correspondence which associates to a tournament T defined on X a nonempty subset $S(T)$ of X, which is steady by tournament isomorphism (if the names of the candidates change, the names of the winners change accordingly) and which selects the Condorcet winner of T when there exists such a candidate. In other words, S must fulfil the following three conditions:

1. for any tournament T defined on the set X of candidates, we have $\emptyset \neq S(T) \subseteq X$;
2. for any isomorphism φ, we have $S \circ \varphi = \varphi \circ S$;
3. if T admits a Condorcet winner C, $S(T) = \{C\}$.

Among the different tournament solutions, the one designed by A.H. Copeland [6] is easy to compute (more precisely, it is polynomial, see for instance [3] or [7]). To define it, let $s(x)$ denote, for any candidate x, the number of candidates defeated by x, *i.e.* the number of candidates y with $m_{xy} > m_{yx}$; $s(x)$ is called the *Copeland score* of x; it is equivalently the number of candidates y with xTy. Then a *Copeland winner* is any candidate with a maximum Copeland score. We may illustrate this thanks to the previous example.

Example: For the previous example, the Copeland scores s_{Ex} of the four candidates are $s_{Ex}(a) = s_{Ex}(b) = 2$, $s_{Ex}(c) = s_{Ex}(d) = 1$. Thus a and b are the Copeland winners of the tournament T_{Ex}.

In his paper [12], H. Moulin studies several properties of Copeland's solution, even if he advocated the choice of other tournament solutions. Copeland's solution is so simple and popular that it is usually used as a reference to measure the disparity between tournament solutions, under the name of *Copeland value of a solution* [8] – see also [9].

Another way to obviate the lack of Condorcet winner is to alter the set of the preferences of the voters in order to obtain a Condorcet winner. Such an approach was suggested by H. P. Young in 1977 [15]. In [15], H. P. Young suggested to remove the minimum number of voters so that a Condorcet winner appears. For each candidate x, we define the *Young score* $Y(x)$ of x as the minimum number of voters whose simultaneous removals allow x to become a Condorcet winner. Any candidate with a minimum Young score is a *Young winner*. We illustrate this once again thanks to the previous example.

Example: The Young scores Y_{Ex} of the four candidates of the previous example are: $Y_{Ex}(a) = 4$, $Y_{Ex}(b) = 4$, $Y_{Ex}(c) = 2$, $Y_{Ex}(d) = 4$. To show this, note that removing a voter decreases m_{xy} by 0 (if y is preferred to x by the considered voter) or by 1 (if x is preferred to y by the considered voter). Thus, if we consider two candidates x and y with $m_{yx} - m_{xy} \geq 0$, we must remove at least $m_{yx} - m_{xy} + 1$ voters if we want to obtain a new difference $m_{xy} - m_{yx}$ which is strictly positive. Applied to our example, this argument leads to $Y_{Ex}(a) \geq 4$ (because $m_{da} - m_{ad} = 3$), $Y_{Ex}(b) \geq 4$ (because $m_{ab} - m_{ba} = 3$), $Y_{Ex}(c) \geq 2$ (because $m_{bc} - m_{cb} = 1$), $Y_{Ex}(d) \geq 4$ (because $m_{cd} - m_{dc} = 3$). Then it is easy, in our example, to show that these inequalities are in fact equalities. For instance, to make c become a Condorcet winner, it is sufficient to remove the first two voters (other choices are possible). Thus, c is the only Young winner of T_{Ex} (observe that c is not a Copeland winner of T_{Ex}...).

III. LINK BETWEEN COPELAND'S SOLUTION AND THE REMOVAL OF CANDIDATES

Instead of removing voters from the election, as in Young's method, we can wonder how many candidates it would be necessary to remove in order to obtain a new election admitting a Condorcet winner. More precisely, for each candidate x, we define the *removal score* of x, $r(x)$, as the minimum number of candidates that must be removed so that x becomes a Condorcet winner. Thanks to these scores, we may define the winner of the original election as any candidate x with a minimum score $r(x)$. Let us call this procedure the *candidates-removal procedure*.

Note that other voting methods are based on the removal of candidates in order to find a winner. For instance, the procedure called *single transferable vote* (STV, also known as *preferential voting* or *preference voting*, or still as *instant-runoff voting*) iteratively removes the candidate who is the least often ranked at the first position until there is a candidate who gained at least $(m + 1)/2$ votes, where m still denotes the number of voters. Similarly, E.J. Nanson [13] and J.M. Baldwin [1] designed methods based on Borda's method [2], and consisting in eliminating iteratively candidates with low Borda scores until only one candidate remains, who is then the winner. But, to my knowledge, nobody suggested the procedure depicted above.

We show now that the candidates-removal procedure provides the same winners as Copeland's solution (for the example, we have $r(a) = 1$, $r(b) = 1$, $r(c) = 2$, $r(d) = 2$: a and b are the winners according to the candidates-removal procedure; as seen above, they are the Copeland winners as well). For this, we assume that we deal with m voters who must choose between n candidates; X denotes this set of candidates. We also assume that the preferences of the voters are linear orders defined on X. Moreover, with the same notation as above, we assume that there is no tie: $\forall x \in X$, $\forall y \in X$ with $x \neq y$, $m_{xy} \neq m_{yx}$. The result stated in the following theorem is linked to the usual axiom of independence of irrelevant candidates: the collective preference between candidates x and y must depend only on the individual preferences between x and y. In other words, the collective preference between x and y must remain the same as long as the individual preferences between x and y do not change. It is what happens when a candidate z is removed, leaving the quantities m_{xy} and m_{yx} unchanged for $x \neq z$ and $y \neq z$.

Theorem

For any election fulfilling the previous hypotheses, any winner according to the candidates-removal procedure is a Copeland winner of the majority tournament and conversely.

Proof

Let x be a candidate. Let $D(x)$ be the set of candidates who defeat x: $\forall y \in D(x)$, $m_{xy} < m_{yx}$. For $\notin \{x, y\}$, the removal of z does not change the values of m_{xy} and m_{xy}. Thus, when we apply the candidates-removal procedure, the only way to transform x into a Condorcet winner is to remove all the candidates belonging to $D(x)$. Conversely, removing all the elements of $D(x)$ suffices to transform x into a Condorcet winner of the election induced by the remaining candidates. Hence the relation $r(x) = |D(x)|$, where $r(x)$ still denotes the removal score of x. On the other hand, we have the following relation between the Copeland score $s(x)$ and $r(x)$: $s(x) = n - r(x) - 1$. Indeed, any candidate y other than x either defeats x, and then y belongs to $D(x)$, or is defeated by x, and then y contributes for 1 to the Copeland score of x. Hence the relations $|D(x)| + s(x) = r(x) + s(x) = n - 1$.

So, minimizing $r(x)$ in order to obtain a winner according to the candidates-removal procedure is the same as maximizing $s(x)$, which provides a Copeland winner and, conversely, maximizing $s(x)$ in order to obtain a Copeland winner is the same as minimizing $r(x)$, which provides a winner according to the candidates-removal procedure. ■

Though its proof is very easy, this result answers a natural question with respect to Young's procedure (what happens if we replace the removal of voters by the removal of candidates?) and provides a new (to my knowledge) property of Copeland's solution. With this respect, it seems to me that it deserves to be noted.

Notes and References

1. Research supported by the project ANR-14-CE24-0007-01 "CoCoRICo-CoDec".

2. Note that, according to I. McLean, R. Llull already promoted this method: "Ramon Llull proposed a Condorcet method (should we now call it a Lull method ?) of pairwise comparisons" [10].

3. According to I. McLean, H. Lorrey and J. Colomer, Copeland's method was already considered by R. Llull: "Ramon Llull (ca 1232-1316) (…) made contributions which had been believed to be centuries more recent. Llull promotes the method of pairwise comparison, and proposes the Copeland rule to select a winner." [11].

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